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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.

2. To supply an additional medium for the publication of expository mathematical articles.

3. To promote more scientific methods of teaching mathematics.

4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Mathematics, Servant to Humanity

The proper study of mathematics leads easily in time to deeply preoccupied states of mind that often make havoc of human bonds and human conventions. No other research requires quite so much isolation from wide human contacts as scientific research, and no kind of scientific research demands at times such utter detachment from one's fellows as research in mathematics. Newton would leave his guests to pore over an unsolved problem. Gauss went to but one meeting of scientists in all his life, "was morose and little communicative".* Lagrange was "timid in conversation".* Laplace, on becoming distinguished, "held himself aloof from his relatives and from those who had assisted him".† Archimedes forgot his state of undress as he ran along the streets of Syracuse shouting aloud a discovery.*

Because so many of its truths have been and tend to be found by minds working in relative solitude, the pursuit of mathematics may be called a solitary one. But the solitude relates only to the general mass of humanity, not to the special portion of it devoted to mathematics; for mathematicians, however far apart or however different in nationality, very often become closely associated, especially if they pursue the same research. Indeed, the more thorough is an individual's research the more multiplied are apt to be his contacts with others who have worked or are working in the same field. But the instances of human detachment cited above from the group of major immortals in mathematics index a detachment that has marked in some degree nearly all the tribe of mathematical workers, a detachment more or less invariant through all the advances of time.

*Cajori's *History of Mathematics.*
†Ball's *History of Mathematics.*

In the past, high mathematical achievement has generally been assumed to be rooted in high native intellectual ability—an ability sufficiently marked to set the possessor of it definitely apart from the majority of his fellows. If this assumption should in reality be partly or even wholly false, as long as it is believed to be true by a general public, the consequences must be the same: Every mathematically active mind will be regarded by the average person as having interests measurably detached from those of the average group of his human fellows.

Thus, broadly speaking, two forces may be more or less clearly perceived to have influenced the mathematician's place in human society; first, a force proportional to the amount of the individual's activity and operating to widen the distance between the centre of his own interests and that of general human interests; second, a force operating to strengthen bonds of association between the individual mathematician and the small group of his fellow mathematicians who are motivated by research interests in common with his own. It may be convenient to refer to these forces as centrifugal and centripetal forces, respectively. They are not independent, since generally the more the individual is influenced by his research the stronger is his tendency to be removed from the interest centres of a general humanity. Facts verifying this could be cited almost without limit from mathematical history.

But just as the two distinct foci of a conic correlate unerringly to a single centre of it, so must the foci considered in this analysis, namely, on the one hand centres of the vast world of red-blooded human interests, and, on the other hand, centres of unnumbered domains of mathematical research, correlate somewhere to a single centre of centres. One may ask if this is provable. Not in its absoluteness as no absolute

can be proven. But the inductive evidences are overwhelming. Though preoccupied makers of mathematics have drifted in their creative dreams from familiar human moorings, mathematics itself, with expanding activity, throughout all eras has been serving humanity. Called by Gauss "Queen of the Sciences", in this time when its uses are multiplying almost overnight, with much more accuracy, certainly with more significance, should it be called the never-aging servant to humanity!

So, springing out of the very detachments, solitudes and eccentricities of the creators of mathematical science, born, as it were, of the travailing pains of genius, has come a centralizing force, namely, mathematics itself. The centre of the vast cosmos of general human interests is under its steadily growing influence. Most pressing problems of human need are being solved by its help, and the more closely drawn together are the individuals of human society by apparent ultra-rational forces, the more pervasive and delicate seem to become the ministrations of mathematics to the needs of man.

In the May issue of National Mathematics Magazine we shall analyze the phase of the human-service functions of mathematics that relates to present problems of school and college programs.

S. T. SANDERS.

A Trisector

By ROBERT C. YATES
University of Maryland

Following that remarkable discovery of Peaucellier and Lipkin for converting circular into linear motion, the study of linkages became an exciting fashion throughout Europe, attracting many of the ablest mathematical minds of the age. In the year 1875 this fast growing enthusiasm reached its maximum height with the publication of more than forty papers on the subject! The subsequent drop in interest, almost as remarkable as the rise, was no doubt due entirely to two significant things. First, Kempe proved that it was possible to describe any algebraic curve by means of a pure linkage; second, Sylvester, who nursed the subject into full bloom, left for America to establish a department at Johns Hopkins and to have his own interest forcibly set in new directions by a persistent student.

During the development of linkage mechanisms no one would have considered incorporating any sort of sliding device since that would have destroyed the underlying principle of the "compound compass". From the point of view of the machinist, slides were objectionable because of the difficulty of tooling metal in decent fashion and a machine which depended upon pin joints entirely was smooth and free in its movement.

It thus seems a little strange that there should appear in the report of the congress at Nantes* for 1875 an article in which M. Laisant proposed a contrivance for trisecting the angle. This mechanism contained a sliding device but even at that it did not possess the compensating virtue of having a simple construction. The essential idea, more than likely prompted by the appearance of the Peaucellier cell, is contained in Fig. 1.

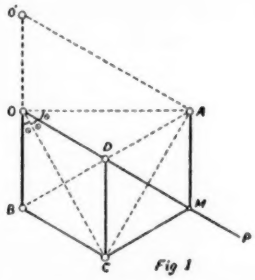


Fig 1

*Laisant: "Note sur un Compas Trisecteur", Compte rendu de la 4e session de l'Assoc. franc. pour l'avanc. des sciences (Congrès de Nantes), 1875, pp. 161-163.

The bar OP is of indefinite length upon which the joint M is forced to slide by means of a groove. The other bars, OB, BC, CD, DA, AM, and MC are all of equal length with D attached to OP so that $OD = OB =$ etc. Now since $OBCD$ is a rhombus, the diagonal OC bisects the angle BOD. Furthermore, since M is constrained to OP, triangles AOD and DOC are homologous with equal angles at O. Thus the arbitrary angle AOB is trisected by the lines OC and OP.

However, by the addition of a single bar and the rearrangement of some others, we may discard the undesirable slide and have a pure linkage containing only joints as shown in Fig. 2.

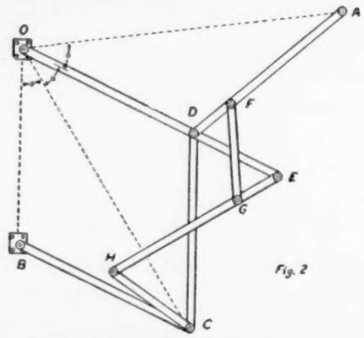


Fig. 2

As in Fig. 1, we take $OB = BC = CD = DO = DA = a$, extending the bar OD to E for any convenient length b. The remaining bars are selected and attached so that $HE = DC = a$, $FG = DE = b$, $DF = GE = c$. Since the purpose of the slide M of Fig. 1 is to have OM bisect the angle ADC then we must have:*

$$b^2 = ac$$

so that angle $ADE =$ angle EDC. In all positions of the mechanism then OE and OC trisect the angle AOB.

An additional feature of the mechanism is worth noticing. Let us attach O and B to a base plane and consider the locus of the point A (either figure). The line joining A and C is at all times perpendicular to OD and thus to BC. It is then the tangent to the circle described by C. If now we join A to the point O' (Fig. 1), taken upon the line OB at a distance a from O, this line will always be parallel to OD and perpendicular to AC. It is evident then that A traces the pedal curve with respect to the pole O' of the circle described by C. It is well known that this is the Limaçon of Pascal, the curve here being symmetrical about the line $O'OB$.

*See "An Ellipsograph", Nat. Math. Mag., Feb. 1938, pp. 213-215.

A Graphical Solution of the Cubic

By H. B. CURTIS
Lake Forest College

There is a fascination in attempting an original solution of the cubic, especially a graphical solution. The literature* is full of such solutions from Tartaglia to the present time. There is no claim that the following solution is new, but it is original with the writer, and on account of its simplicity it may be of interest to the beginner. The cubic itself need not be graphed, for all solutions are made to depend on the graph of the cubical parabola $y=x^3$. Of course the case of imaginary roots is the one of most interest.

Assume the given cubic in the form

$$x^3+3Hx+G=0 \qquad (A)$$

where G, for convenience, will be taken negative.†

In the figure choose convenient units for x and y and with a slope equal to $-3H$ and a $y-$intercept

$$(1) \qquad OC=-G$$

draw the straight line CP cutting $y=x^3$ in P. The abscissa of P is a real root of the cubic (A). If CP intersects the curve in additional points, the abscissas of these points are the other real roots. Otherwise, the other roots are imaginary. The absolute value of the imaginary parts may be found graphically by the following construction.

Draw PT tangent to $y=x^3$ at T, intersecting the $y-$axis at B. Vertically below B, construct A, making $AB=BC$. Now draw PA cutting the curve in D and E. Then the distance of D or E from a vertical line through T, that is, SD or EF is the absolute value of the imaginary parts of the imaginary roots. The abscissa of T is the real part.

*Bérard: *Opuscules Mathématique et Methodes nouvelles pour déterminer les racines des équations numériques*, p. 33; Monge: *Correspondence sur l'Ecole Polytechnique*, vol. 3, pp. 201-203; *The American Mathematical Monthly*, vol. 27 (1920), pp. 203-204; vol. 28 (1921), pp. 415-423; vol. 42 (1935), pp. 383-384.

†If $G>0$, we consider the equation whose roots are the negatives of the roots of the given equation.

Proof. Let the roots of (A) be 2α, $-\alpha+\beta i$, $-\alpha-\beta i$. Factoring $x-2\alpha$ out of the left side of (A), the imaginary roots must satisfy the quadratic

$$x^2+2\alpha x+4\alpha^2+3H=0$$

and thus are $-\alpha\pm\sqrt{-(3\alpha^2+3H)}$. Whence $\beta=\sqrt{3\alpha^2+3H}$.

Now let a new straight line through P cut the curve in two other real points whose abscissas are $-\alpha+\beta$ and $-\alpha-\beta$,* and intersect the y-axis in A. Then $OA=-6H\alpha-4\alpha^3$, for this value is the product of all three abscissas, exactly as $-G$ is the product of all the roots of (A). Since 2α is a root of (A),

$$OA=-6H\alpha-4\alpha^3=G+4\alpha^3 \tag{2}$$

Again, since the line PT, which passes through $(2\alpha,\ 8\alpha^3)$ has as slope $3\alpha^2$,

$$OB=2\alpha^3 \tag{3}$$

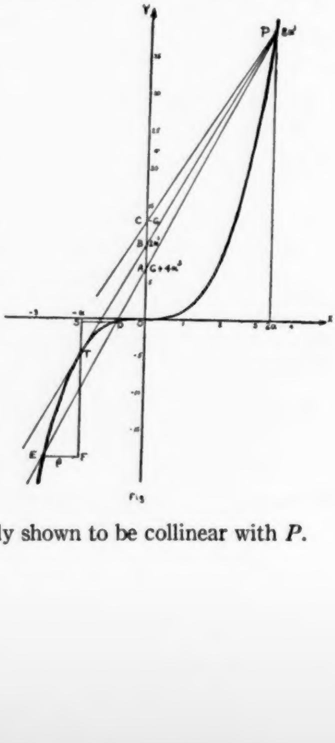
From the values of the segments OC, OA, OB as given by (1), (2), (3) respectively, the obvious identity

$$2\alpha^3-(G+4\alpha^3)=-G-2\alpha^3$$

gives $$AB=BC$$

This completes the proof.

The figure shows the graphic solution of the cubic $4x^3-31x-52=0$.



*These points are easily shown to be collinear with P.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Sidelights on the Cardan-Tartaglia Controversy

By MARTIN A. NORDGAARD
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I.

There is quite a difference in the frame of mind which comes with the answer to a problem only vaguely defined and lying in an uncharted field, like the invention of the differential calculus, or with a discovery that comes undivined like a flash of lightning from some human mind, like the invention of logarithms,—and the reaction that greets the answer to a problem posed to the world for centuries when that answer arrives, two thousand years in the coming.

The solution of the cubic had presented itself to the human mind as an intellectual problem already in the fifth century B. C.; it became a scientific need in Archimedes' calculation on floating bodies in the third century B. C.; it confronted the Arab astronomers in the Middle Ages. And now it was solved! The first of "the three unsolved problems of antiquity" to be solved.

It produced a great impression. How great, one can gauge from the fact that all respectable texts on algebra for the next 200 years gave long chapters and discussions to the cubic equation. The influence of the discovery must be gauged not only by its mathematical fruitfulness, which after all did not prove to be so very great, but by the stimulus it gave to study, the courage it gave the human mind to soar into the unknown and "make the impossible possible."

The main events leading up to the discovery of a general solution of the cubic equation and the ensuing controversy are given in the various histories of mathematics. But there are illuminating sidelights in this unique controversy,—documentary, anecdotal, biographical,—which do not lend themselves to recording in a well-balanced

history of mathematics but which are of absorbing interest to the members of the guild of mathematicians. There are the many source materials, for one thing; from some of these we shall quote extracts. There is the language and symbolism, or lack of it, of the algebra prior to Vieta, Stevin, and Descartes. And then there is the exposition of the status of algebraic theory before the monumental works of Cardan and Tartaglia.

The 16th century custom of scientific "duels" and public disputations were a joint inheritance from the philosophical disputations of the Schoolmen and the tournaments of the knights. A chief canon of combat was that no one should propose a question or problem that he himself could not solve. The outward forms were modeled somewhat after the contests of arms,—challenge, response, witnesses, judges, keeper of the stakes, etc.

Public challenges were given, not only for acquiring glory and prestige, but also for making a living. The vanquished, honor lost, had no more pupils; while the victor, heralded and fêted, would be called to various cities to teach and lecture. Consequently, many inventors guarded their secrets. There must have been many discoveries lost to the world due to this custom. Tartaglia himself died while still writing on his algebra and before reaching his contemplated climax on his solution of the cubic; and except for the premature publication of it by Cardan and Tartaglia's accusation in the ***Quesiti*** his solution might have died with him.

II.

The *Dramatis Personae* of the celebrated controversy were five: Zuanne de Tonini da Coi, Antonio Maria Fior, Girolamo Cardano, Nicolo Tartaglia, and Ludovico Ferrari. The time: 1530 to 1548. Place: Pavia, Padua, Bologna, Milano, Brescia, Venice, the centers of art and learning in Renaissance Italy.

The first two were minor characters and little is known about them outside their connection with this controversy; they were messengers, links, as it were, to bring about action between the other three. ***Zuanne da Coi*** (sometimes called Giovanni dal Colle) was a teacher in Brescia interested in mathematics from the standpoint of problem solving. ***Antonio Maria Fior*** (sometimes written Floridus and Del Fiore) flitted about from place to place, causing battle and disturbance; but History will thank him for it. He was an arithmetician, having according to reports no theory knowledge in algebra. He had been a pupil of Scipio Ferro, of whom more later.

Nicolo Tartaglia was born at Brescia in 1506, died at Venice in 1557. He came from a very poor family, was left fatherless at the age of six, and had only two weeks of formal schooling; but by self-education his powerful mind mastered both the classics and the then known mathematics. He taught mathematics in Verona, Vicenza, Brescia, and, from 1534 or 1535 until his death in 1557, in Venice. His principal mathematical works are: *Nova Scienza* (1557), where he is the first one to discuss the problems of gunnery and fortification mathematically; (2) *Quesiti ed invenzioni diverse* (1546), in nine books, of which the last one deals with algebra; (3) *General Trattato di numeri, e misure* in two volumes (the first published in 1556, the second in 1560) including an arithmetic, a treatise on numbers, and his work on algebra.

Girolamo Cardano (Hieronymus Cardanus or Jerome Cardan) was born at Pavia in 1501, died in Rome in 1576. He received a good university education in Pavia and Padua, having equal zest for medicine and mathematics. Between 1524 and 1550 he taught and practiced medicine, much of the time in Milano; in the same period he studied mathematics assiduously and published many important works. In 1562 he became a university professor at Bologna and in 1570 he moved to Rome to become astrologer to the Pope. He wrote voluminously on many subjects, but in mathematics we mention these: (1) *Practicae Arithmeticae* (1539); (2) *De Regula Aliza* (1540); (3) *Ars Magna* (1545), the first systematic work in algebra up to that time, a text that helped to clarify the principles of algebra and lift the subject out of mere equational problem solving into a theory of equations.

Ludovico Ferrari was born at Bologna in 1522 and died there in 1565. His parents were poor and he came to Cardan's house as an errand boy. He was later allowed to listen to his master's lectures and before long became Cardan's most brilliant pupil. For all his moral lapses and irascible temper, Ferrari was ever loyal to his protector; in fact, looked upon himself as owing his very being to Cardan, designating himself as "suo creato." As far as we know he never published anything independently. What he discovered he let Cardan incorporate into the *Ars Magna*. One of his discoveries was a general solution of the biquadratic equation. For he was able, by using the solution of a cubic already discovered by Tartaglia, to solve the question proposed by Da Coi, namely $x^4+6x^2+36=60x$, succeeding where both Tartaglia and Cardan had failed. Ferrari was only twenty-three years old when the *Ars Magna* was published.

In reading works on algebra from this period the reader must try to divorce from his consciousness many of the ideas and forms he

has associated with algebra. He will remember that in 1500 there were no imaginary numbers; they did at times make unwelcomed appearances but were not legitimized. Negative numbers did not have operational status and $x^3=px+q$ had a different solution from that of $x^3+px=q$, for instance. The symbols $+$, $-$, $=$, in our sense, did not exist, and our words "plus" and "minus" were not conventionalized. The unknown was variously called *thing, side, cos, res.* Thus Tartaglia's equation ("capitolo") $x^3+3x^2=5$ was "a cube and three censi are equal to five."

Besides Cardan's *Ars Magna* and Tartaglia's *Quesiti* and *General Trattato* we have as source material the six *Cartelli* (letters of challenge) of Ferrari and the six *Risposti* (responses) of Tartaglia. These were sent as printed pamphlets to the mathematicians of Italy.

Being literature of a day it is a wonder that all the twelve bulletins have come down to us. As one might expect, they have their own exciting history. In 1844 Prof. Silvestro Gherhardi owned a volume containing the six Cartelli of Ferrari and the first five Risposti of Tartaglia. In 1848, after a four years' search in all the libraries and old book-stores of the various cities of Italy, Gherhardi finally laid his hands on the missing 6th Risposta in an old book shop in Bologna, and it is the only copy of this Risposta found so far.* Previously all that was known of Tartaglia's 6th letter were citations from Bombelli (1572) and writers living later than Tartaglia by 200 years. In 1858 Gherhardi, meeting with political vicissitudes and exile and in need of money, sold his copy to Libri of London, it being "clipped" on to the other eleven. But firstGherardi was permitted to make an exact copy of the letters "by the hand of Benaducci di Foligno." And that was fortunate; for the copy sent to Libri was lost. So it has been by a slender thread that the last of the twelve letters has reached us. In 1876 the twelve letters were "collected, autographed, and published" by Enrico Giordani in a limited edition of 212 copies under the title *I sei cartelli di matematica disfida di Ludovico Ferrari con sei contro-cartelli in risposta di Nicolo Tartaglia.*†

An additional word concerning Tartaglia's *Quesiti ed invenzioni diverse.* It consists of short, sprightly accounts in dialogue form of problems he discussed with or solved for various people, the first *Quesiti* dated 1521, the last, 1541. Quoting conversations and letters, citing dates, places, and names of interlocutors, many of whom were still living, the book has strong documentary secureness. It is charmingly written, besides. The last of the nine books, comprising 42 *quesiti,* deals with algebra.

**Cartelli* and *Risposta*: Introduction, pp. 9, 12, 15.
†Hereafter referred to as *Cartelli* and *Risposti.*

III.

The first to give a general solution for a cubic equation was neither Cardan nor Tartaglia. That honor belongs to Scipio del Ferro, professor of mathematics at Bologna.

As late as 1494 Luca Pacioli in his authority-carrying *Summa* had set forth these types of equations as not yet being solved:

$$n = ax + bx^3$$

$$n = ax^2 + bx^3$$

$$n = ax^3 + bx^4.$$

And he intimated that their solutions might not be possible. However, the first of these was solved by Ferro of Bologna.

About all we know of Scipio Ferro is that he was born at Bologna about 1465 and died there in 1526, that he was professor at the University of Bologna from 1496 to 1526, that he had a general solution for $x^3 + px = q$, and that he confided his method to his pupil Antonio Maria Fior. We donot know whether he had derived it himself or found it in an Arab work; whether it was an empirical formula or was the product demonstration. What writings he left must have come into the hands of of his son-in-law Annibale della Nave, who succeeded him in his professorship (1526-1560). But no such writings are extant. Both Cardan and Tartaglia refer to the solution Fior received from Ferro, Tartaglia placing it about 1506, Cardan placing it at about 1514.* It was probably even later.

IV.

Curiously enough, the one who seems to have set the wheels in motion for the final onslaught on the cubic equation was a man of meagre mathematical attainment but of much physical mobility. It was Zuanne de Tonini da Coi. Teaching in Brescia he had, of course, heard of the work of Nicolo of Brescia, now of Verona. In 1530, as one Brixellite to another, with more courage than prudence he proposed to Tartaglia two problems which reduced to solving the equations $x^3 + 3x^2 = 5$ and $x^3 + 6x^2 + 8x = 1000$.

This, the opening chapter in the history of the exciting discovery, is described by Tartaglia in his *Quesiti,*† namely in Quesito XIV. There for the first time we learn that Tartaglia (at this time only 24 years of age) had been dabbling with the cubic.

**Ars Magna*, Nürnberg, 1545, Ch. XI; *Quesito* XXV.

†*Quesiti et inventioni diverse de Nicolo Tartaglia.* At the press of the author, 1554.

It will give the reader a little of the flavor of the period and give him a peek into one of the interesting books of mathematics to read Tartaglia's own first reference to the attack on the cubic equation.

"QUESITO XIV,

which was propsed to me at Verona by one Maestro Zuanne de Tonini da Coi, who has a school in Brescia, and was brought to me by Messer Antonio da Cellatico in the year 1530.

Maestro Zuanne.—Find a number which multiplied by its root increased by three equals five. Similarly find three numbers such that the second is greater by two than the first and the third is greater by two than the second and where the product of the three is 1000.

N.—M. Zuanne, you have sent me these two questions of yours as something impossible to solve or at least as not being known by me; because, proceeding by algebra, the first leads to the operation on a cube and 3 censi equal to 5, and the second on a cube and 6 censi and 8 cose equal to 1000. [That is, $x^3+3x^2=5$; $x^3+6x^2+8x=1000$]. By F. Luca and others these equations have up to now been considered to be impossible of solution by a general rule. You believe that with such questions you can place yourself above me, making it appear that you are a great mathematician. I have heard that you do this towards all the professors of this science in Brescia, so that they for fear of these your questions do not dare to talk with you; and perhaps they know more about this science than you. . . . .

M. Z.—I understand as much as you have written to me and that you think such cases are impossible, ...

N.—I do not say such cases are impossible. In fact, for the first case, that of the cube and the censi equal to a number, I am convinced I have found the general rule, but for the present I want to keep it secret for several reasons. For the second, however, that of the cube and censi and cose equal to a number, I confess I have not up till now been able to find a general rule; but with that I do not want to say it is impossible to find one simply because it has not been found up to the present . However, I am willing to wager you 10 ducats against 5 that you are not able to solve with a general rule the two questions that you have proposed to me. And that is something for which you should blush, to propose to others what you yourself do not understand, and to pretend to understand in order to have the reputation of being something great."

That ends the first encounter.

V.

We now go back a ways to the afore-mentioned pupil of Scipio del Ferro, Antonio Maria Fior, sometime of Venice. He seems to have turned Ferro's formula to account in the popular mathematical contests then in vogue. Hearing of Tartaglia's claim to solving some cubic, possibly publicized through Da Coi, and thinking Tartaglia an impostor and himself knowing Ferro's solution of $x^3+px=q$, he challenged the latter to a contest. It was set for February 22, 1535. Tartaglia, knowing Fior was only an arithmetician, gave the contest little thought at first. But when he heard that "a great master" "30 years ago" had communicated to him the solution of a cubic, he became worried and set himself to study the equation $x^3+px=q$. (He already had solved $x^3+px^2=q$). On February 14, eight days before the date set for delivering the solutions to the notary who kept the stakes, he found the solution of $x^3+px=q$; and on the next day he also found the solution of $x^3=px+q$.*

Each had challenged the other with thirty questions. As Tartaglia had suspected, all Fior's problems reduced to the form $x^3+px=q$, and he solved them all in two hours. It almost seemed wicked of Tartaglia, for he had constructed problems such that most of them led to the solution of $x^3+px^2=q$ and Fior could not answer a one of them. "I waived the stake and took the honor," says Tartaglia.

Thus ended the second encounter.

We read of these things in the histories. But our modes of life and thinking, our physical environment, are so removed from the 16th century Italy that it is hard for us to reconstruct the tenseness and excitement that accompanied these contests. Honor, gold, and the instinct of game were powerfully present. The questions themselves—the instruments of combat—what did they look like? The histories tell us about $x^3+px=q$. That seems so general and colorless. And then there were no $x^3+px=q$. There were "cube and cose equal to a number" and similar expressions. The challenges did not come in that form either,—they came as problems. And since this is a sidelight, we shall see what they are. And, gentle reader, so as to be along in spirit with the tense partisans of that February 22, 1535, solve one or two; you are along in the opening skirmish of the famous "Battle of the Cubic" of the 16th century.

These were the questions submitted by Fior for February 22, 1535:†

(1) Find the number which added to its cube root gives 6.

*Quesito XXV.
†Quesito XXXI.

(2) Find two numbers in double proportion $[x,2x]$ such that if the square of the larger is multiplied by the lesser and to the product is added the sum of the numbers, the result is 40.

(3) Find a number which added to its cube gives 5.

(4) Find three numbers in triple proportion $[x,3x,9x]$ such that if the square of the smallest is multiplied by the largest and the product be added to the mean number, the result is 7.

(5) Two men were in partnership, and between them they invested a capital of 900 ducats, the capital of the first being the cube root of the capital of the second. What is the part of each?

(6) Two men together gain 100 ducats. The gain of the first is the cube root of the gain of the second. What is the gain of each?

(7) Find a number which added to twice its cube root gives 13.

(8) Find a number which added to three times its cube root gives 15.

(9) Find a number which added to four times its cube root gives 17.

(10) Divide fourteen into two parts such that one is the cube root of the other.

(11) Divide twenty into two parts such that one is the cube root of the other.

(12) A jeweler buys a diamond and a ruby for 2000 ducats. The price of the ruby is the cube root of the price of the diamond. Required the value of the ruby.

(13) A Jew furnishes capital on the condition that at the end of the year he shall have as interest the cube root of the capital. At the end of the year the Jew receives 800 ducats, as capital and interest. What is the capital?

(14) Divide thirteen into two parts such that the product of these parts shall equal the square of the smallest part multiplied by the same.

(15) A person buys a sapphire for 500 ducats and gains the cube root of the capital invested. What was his gain?

(16-30) Deal with lines, triangles, and various equilateral polygons with sides so divided as to become problems of dividing 7, 12, 9, 25, 26, 28, 27, 29, 34, 12, 100, 140, 300, 810, 700 each into two parts such that one is the cube root of the other.

As we see, all these reduce to the form $x^3+px=q$.

Of Tartaglia's 30 challenge questions to Fior we have record of only the first four. These follow:*

(1) Find an irrational quantity such that when it is multiplied by its square root augmented by 4, the result is a given rational number.

(2) Find an irrational quantity such that when it is multiplied by its square root diminished by 30, the result is a given rational number.

(3) Find an irrational quantity such that when to it is added four times its cube root, the result is thirteen.

(4) Find an irrational quantity such that when from it one subtracts its cube root, the result is 10.

These problems resolve themselves into solving for the irrational quantity x in $x^3+mx^2=n$; $m^2x^2=x^3+n$; $x^3+mx=n$; $x^3=mx+n$.

VI.

It is the ever-moving Da Coi again who brings in the next important personage in these events, Girolamo Cardano. For after his interview with Tartaglia he leaves Brescia and moves to Milano. There he meets Cardan who engages him to instruct one of his classes. Da Coi tells him about Tartaglia and his discovery. Cardan, at this time preparing material for his ambitious work, *Ars Magna,* was much interested in the mathematical duel of Tartaglia and Fior. He therefore sends as messenger Zuan Antonio de Bassano, a book seller, to Tartaglia to inquire about his invention. The atmosphere of the time and the temperament of the principals are well sketched by Tartaglia, under date of January 2, 1539, in

"Quesito XXXI. *Fatto da M. Zuanantonio libraro, per nome d' un Messer Hieronimo Cardano, Medico et delle Mathematice lettor publico in Milano, adi. 2. Genaro, 1539.*"

Zuantonio.—Messer Nicolo, I have been directed to you by a certain man, a good physician in Milano called Messer Hieronimo Cardano, who is a great mathematician and gives public lectures on Euclid in Milano; at present he has a work in press on the art of arithmetic, geometry, and algebra, which will be a beautiful thing.† He has heard of the contest you had with Maestro Antoniomaria Fiore and how

*Quesito XXV.

†Al presente sa stapare una sua opera in la practica Arithmetica et Geometria et in Algebra che sara una bella cosa.

you agreed to prepare 30 cases or questions each, and that you did that. And his Excellency has heard that all the 30 questions proposed to you by Maestro Antoniomaria led you by algebra to an equation of the cosa and the cube equal to a number (*che ui conduceano in Algebra in un capitolo di cosa e cubo equal a numero*), and that you found a general rule for such an equation and that by the power of this invention you solved in the space of two hours all the 30 cases he proposed to you. On this account his Excellency begs that you would be so kind as to send him this rule that you have invented; and if it pleases you he will insert it in his forth-coming book under your name.

N.—Tell his Excellency that he must pardon me; that when my invention is to be published it will be in my own work. His Excellency must excuse me.

Z.—If you do not want to impart your invention to him, his Excellency ordered me to ask you at least to let him have the 30 above-mentioned cases which were proposed to you together with their solutions. [meaning the results, not the rule obtaining them.]

N.—Not even that can be. For whenever his Excellency observes one of these cases and its solution he will get to understand the rule that I found. And by means of this one rule many others dealing with this subject can be derived." So far, Tartaglia,

After this second rebuff Zuanantonio proposes seven problems leading to these equations:

(1) $2x^3+2x^2+2x+2=10$

(2) $2x^3+2x^2+2x+2=10x$

(3) $2x^3+2x=10$

(4) $2x^3+2x^2=10$

(5) $2x^3+2=10x$

(6) $x^4+8x^2+8^2=10x^3$

(7) $x^3+3x^2=21$

Somewhat hotly Tartaglia rejoins: "These questions are from Messer Zuanne da Coi. And from no one else, for I recognize the last two. Two years ago he proposed to me a question like the sixth and I made him own up that he neither understood the problem nor knew the solution. He also proposed one similar to the last one, which involves working in census and cubes equal to a number [that is, $x^2+px^3=q$] and out of my kindness I gave him the solution less than a year ago. For such solutions I found a particular rule applicable to similar problems."

The bookseller maintains the questions are Cardan's, however. And to support his request he praises Cardan's abilities and deftly

mentions his connection with a certain high and rich personage, the Marquis del Vasto, a benefactor who was to publish Cardan's book.

"I do not say his Excellency is not a very learned and capable person", says Tartaglia. "But I say he is not able to solve the seven problems which have been proposed to me to be solved by a general rule."

When the messenger leaves he gives him a copy of Fior's 30 questions but not the solutions.

Cardan's reply, February 12, 1539, is full of bitter insult. You are not at the top of the mountain, you are only at its foot, in the valley, he tells Tartaglia, in substance. It is peculiar that you attribute the seven problems to Da Coi, as if there were no one in Milano able to do such a thing. Da Coi is as young as he says he is; I have known him since before he could count to ten. You said if one of these problems is solved, they all are solved. That is completely wrong. I wager 100 ducats you are not able to reduce them to one, nor to two, nor to three equations. (This is the purest invention of Cardan: Tartaglia had said nothing of the kind. Or else the bookseller had misunderstood him.) Concluding, he proposes two problems. The first, taken from Pacioli but not solved by him, requires to find four numbers in geometric progression whose sum is 10 and whose square sum is 60. The second concerns two men in partnership who gain the cube of the tenth part of their several capitals.

To the first Tartaglia in his restrained reply of February 18 gives an elegant solution.* But he is not cajoled into giving away his secret by solving the second, still keeping to himself his solution of "the cube and the cose equal to a number."

Neither tricks nor insults succeeding, Cardan turns to flattery and praise. So in a letter dated March 13, 1539 he begins:† "Messer Nicolo, mio carissimo." Asks Tartaglia not take his former words up in bad part. Blames it onto Da Coi who had given him a wrong idea of Tartaglia. Now the ungrateful wretch has left Milano unceremoniously and also left the sixty pupils he had secured for him. He ends by inviting Tartaglia to visit him in Milano and says that the Marquis del Vasto is anxious to meet him. (This was probably pure fiction.) He concludes the letter with high priase for the nobleman and warm feelings for "mio carissimo" Tartaglia:

"And so I urge you to come at once, and do not deliberate; for the said Marquis is a remunerator of all virtuosi, so liberal and magnanimous that no one who serves his Excellency in any matter remains

*Quesito XXXIII.
†Quesito XXXIII.

unsatisfied. So do not hesitate to come, and come and live in my house and no otherwheres. May Christ keep you from harm.

March 13, 1539. Hieronimo Cardano, ***Physician.***"

This was the rift in the wall that made Tartaglia's citadel crumble. He accepts the invitation and stays a few days in Cardan's house. Their conversation is recorded in Quesito XXXIV under date March 25, 1539:

C.—It is convenient for us that the Marquis has just left for Vigevano so we can talk about our affairs till he returns. You surely have not been any too obliging in not showing me your solutions of the cube and cose equal to a number that I have so earnestly asked you to do.

T.—I tell you, I am niggardly in this matter, not for the sake of this simple equation only and the things that it has enabled me to find, but for the sake of all the things this equation ought to help me discover in the future. For it is a key that opens up the investigation of a great many other equations. If I were not now occupied with the translation of Euclid (I am already on Book XIII) I would already have discovered a general rule for many other equations. [Then the discusses his plan for his book on algebra.] If I now showed the solution to a speculative mind, like your Excellency, he could easily discover the other solutions and publish them as his own, which would completely spoil my project. [Notice all along the distinction between solutions, answers, and the formula or process that gives the solutions.] This is the reason that has compelled me to be so discourteous toward your Excellency; so much the more since you are about to publish a work on a similar subject and in which work, you wrote me, you would like to insert my invention under my name.

C.—But I wrote you that if that did not meet your approval, I will promise to keep it a secret.

T.—As to that, I just can't believe you.

C.—Then I swear you by the holy Evangels of God and as a true man of honor that I will not only never publish it, but I will write it for myself in code so that no one finding them after my death can understand. If you will now believe me, believe; if not, let it pass.

T.—If I did not believe such an oath, I should certainly be regarded as a man without faith. But I have decided to go to Vigevano to find the Marquis; for I have already been here three days and am tired of waiting. On my return I promise to reveal it all.

C.—If you wish to see the Marquis I will give you a letter so that he may know who you are. But before you go I wish you would show me the rule, as you promised.

Then Tartaglia gives him the solution for $x^3+px=q$ and $x^3+q=px$. Instead of a code Tartaglia gives it in twenty-five lines of poetry, seven tercets followed by a quatrain.* It must have been as good as a code, for in a letter of April 9th† Cardan has trouble with this mathematical poetry. In his reply Tartaglia says it is not $ut=\frac{1}{3}p^3$, but $ut=(\frac{1}{3}p)^3$.

We will just give a taste of this mathematical poetry by quoting one tercet:

> "*Quando che'l cubo con le cose appresso,*
> *Se aggruaglia à qualche numero discreto*
> *Trouan dui altri, differenti in esso.*"

Meaning: If $x^3+px=q$, let $t-u=q$.

The next few lines says: Also let $ut=(\frac{1}{3}p)^3$, then $x=\sqrt[3]{t}-\sqrt[3]{u}$.

Tartaglia must already have begun to feel uneasy, for on leaving Cardan he says to himself: "I will not go to Vigevano. I will go back to Venice, come what may." In the exchange of letters that follow it becomes evident that Cardan is putting his powerful mind to work on Tartaglia's formulae from every angle and soon discerned implications that Tartaglia himself had either not been able to see or was too busy to follow up (he was busy with his translation of Euclid).

The Irreducible Case came up in a letter from Cardan in August, 1539,‡ when Cardan asks: How about

$$x^3=px+q \text{ when } \left(\frac{p}{3}\right)^3 > \left(\frac{q}{2}\right)^2 \text{ as in } x^3=9x+10?$$

Tartaglia saw that Cardan was now making his own investigations and felt none too good about it. He himself could not solve the difficulty, and his answer to Tartaglia lacks frankness. "Has Tartaglia lost spirit maybe from much studying and reading?" banters Cardan§ in his next letter. "If he is sure of understanding the rule he will wager 100 ecus against 25 that he can solve $x^3=12x+20$." Tartaglia did not answer.

On January 5, 1540 came a noteworthy letter from Cardan—noteworthy in the light of what followed.‡‡ Very friendly; not "mio caris-

*For the full Italian version see Cantor's *Geschichte*, (1913), vol. II, pp. 488-9.
†Quesito XXXV.
‡Quesito XXXVIII.
§Quesito XXXIX.
‡‡Risposta I, p. 8.

simo" now, but "quanto fratello". "That devil" Zuanne dal Colle (as Cardan always spelled it) has returned to Milano and caused him no end of grief. Both in his teaching and in other matters.* But, warns Cardan, he has your equation $x^3+px=q$ and $px+q=x^3$, and he boasts that during his sojourn in Venice he had a discussion with Fior and so arrived at what he searched for. Then he tells of various algebraic solutions that Zuanne had solved, giving details. The whole letter looks like a build-up for 1545, to show that the knowledge of the cubic was not Tartaglia's only.

But Tartaglia does not catch the drift. "Cardan has a mind more dull than I thought", he muses. "Zuanne imposes on him when he says that he has the solution of equations. But I do not want to reply. I have no more affection for him than I have for Messer Zuanne. I will leave them to one another. But I can see that he has lost spirit and does not see how things will turn out."

Then all correspondence between these two ceases.

The next five years were quiet, seemingly. Tartaglia busy with his translations of Euclid and Archimedes, holding in abeyance his future work on algebra; Cardan, assisted by the brilliant Ferrari, working on the *Ars Magna*. In 1545 this monumental work appeared from the Nürnberg press of Petreius. In it, with his consent and under his name, was Ferrari's solution of the biquadratic. In it, and with his name, but not with his consent, was Tartaglia's solution of $x^3+px=q$. The solution that was to have been written in code lest the world should get knowledge of it was broadcast on the pages of Cardan's most ambitious work.

It is given as Chapter XI of the *Ars Magna,*† and is prefaced by a statement that Scipio Ferro had first found the solution, that later Tartaglia also invented it and showed the solution, but not the demonstration, to Cardan.‡ Tartaglia had told Cardan he was jealous of the solution of $x^3+px=q$ not so much for the equation itself, but for the work to which it was the key. And true enough, using this key), ten additional chapters on the cubic besides Ferrari's work on the biquadratic, enrich the contents of the *Ars Magna*. How Tartaglia felt when his eyes saw this, we can imagine. Or can we?

Tartaglia's reply to the statement and the act is given in his *Quesiti,*—documented with names, circumstances, dates, places,—published the following year. Cardan never satisfactorily met those

*Quesito XXXX.

†A translation of this chapter, with comments, by R. B. McClenon is found in D. E. Smith's *Source Book of Mathematics*, New York, 1929, pp. 203-6.

‡The edition examined for this article was this same first edition. There were some changes in later editions.

accusations in writing, nor could Tartaglia entice him to meet him in person for a mathematical combat.

VII.

Now comes a most unique spectacle in mathematical history; not as mathematics but as human passions, wickedness, and contrariness.

Cardan did not reply to the accusations of Tartaglia. But Ludovico Ferrari, his grateful pupil, "suo creato", took up the gauntlet for his master. On February 10, 1547, he sent a public challenge to Tartaglia at Venice: a pamphlet with four pages of content and four (!) pages of names of mathematicians in various universities and cities to whom copies of the challenge had been sent, fifty in all. Among these we notice the name of Ferro's successor, "Hannibal dalle Nave"; but neither Da Coi nor Fior.

"Messer Nicolo Tartaglia", it begins, "there has come into my hands a book by you called *Quesiti ed inventioni nuovi,* in the last treatise of which you mention his Excellency Signor Hieronimo Cardano, a physician at Milano, who is at present a public lecturer in medicine at Pavia. And you are not ashamed to say that he is ignorant in mathematics, that he is a dull individual, deserving to have Messer Giovan da Coi placed before him. . . . .
I think you have done this, knowing that Signor Hieronimo has such a great genius that not only in medicine, which is his profession, has he a reputation for ability, but also in mathematics, in which he has at times indulged as a game, to get recreation and enjoyment, and in which he has become so widely successful that without exaggeration he is considered one of the great mathematicians."

Besides a multitude of errors, the challenge continues, Tartaglia has also plagiarized from Jordanus, whose propositions he has placed in the 8th book without citing his name.* Tartaglia has blamed Cardan unjustly; he, who is not worthy to mention Cardan's name (*il quale à pena sete degno di nominare*).† Thereupon he challenges Tartaglia to a public disputation from ancient and modern authors on "Geometry, Arithmetic, and all the disciplines that depend on these, as Astronomy, Music, Cosmography, Perspective, Architecture, and others, ... , and not only on what Latin, Greek and "vulgar" [vernacular, modern] authors write on these subjects, but also on your own inventions."

The time was thirty days; the stake, up to 200 *scudi,* to be decided by Tartaglia. "And in order that this invitation shall not appear too

*Cartello I, p. 2.
†Cartello I, p. 5.

private, I have sent a copy of this writing to each of the gentlemen named below."

Thus begins the fourth part of this celebrated controversy.

Tartaglia replies on February 19, nine days later. Also a printed pamphlet. Equally formal: "From Nicolo Tartaglia of Brescia, Professor of mathematics in Venice, to Messer Ludovico Ferraro, Public Lecturer of Mathematics in Milano." Six pages of compact print, with four witness signatures. But instead of an impressive list of mathematicians at the end, he has a postscript:*

"And in order that this reply of mine shall not appear too private, I have had 1000 copies printed to send them around Italy in general; since I am not acquainted with the cities in Italy or with the universities, where one can buy the friendship and knowledge of experts and scholars, as you do (because, in truth, my experience and acquantance are limited to my study and to my students). For this reason I do not only not have the friendship but not even the acquaintance of these persons."

Therefore, he continues, he will not send his reply to the persons Ferrari names; could not do it even to a certain named person, "for he died two months ago." (Ferrari's challenge had come nine days ago!)

As for a response, he will not meet Ferrari in combat. It is with Cardan he has a quarrel, and when that gentleman is ready, Tartaglia will accept. So he sends a counter-challenge that Cardan and Ferrari on one side and he on the other submit one to the other a list of problems to solve.

Six weeks later, on April 1, comes a second challenge from Ferrari,—*and this time in Latin.* Why change from Italian to Latin? Tartaglia thought he knew.

In Cartello II Ferrari touches upon the solution of the cubic equation.† Tartaglia has taken umbrage at Cardan for publishing his solution of the cubic. What if the published solution was that of a third party? Five years ago, declares Ferrari, in 1542, Cardan and Ferrari were in Bologna and there visited Annibale della Nave, Scipio Ferro's son-in-law, who showed them books written by Scipio; and there was the solution Cardan published. Annibale is alive today and can be called as a witness anytime. (This would be a serious argument, except it is not convincing. Why, since there was no secrecy about it, had Cardan in *Ars Magna* not mentioned Scipio's *writings* and Anibale's part, instead of referring only to Scipio and Fior?)

A large portion of Tartaglia's replies is sparring for objectives. He regularly wants to get into combat with Cardan himself; just as

*Risposta I, p. 8.
*Cartello II, p. 2.

regularly the slippery Ferrari turns him off. Another objective is to have the contest be a list of challenge questions, to be solved in a specific time; Ferrari wants a public disputation in Rome, Florence, Pisa, or Bologna, to be chosen by Tartaglia, and judges to be selected from persons in the city chosen.

Two points taken up in Risposta II evoke our sympathy:

The first concerns the mode of contest. Besides considering the method of challenge lists a better arbiter of ability than public disputations the sender may have had an added reason for his choice. Tartaglia, "the stammerer", had an impediment of speech ever since as a child he was cut by a French soldier in the cathedral massacre at Brescia. In a public disputation he would have a serious handicap engaging the oily Cardan and the brilliant Ferrari.

The second dealt with where and with whom the stake was to be deposited. ***And if only gold was to be used,*** or whether Tartaglia could, for part of the sum, deposit printed copies of the ***Quesiti***. Remembering Tartaglia's worldly circumstances one can here read much between the lines.

Replying to the charge of plagiariam, he says:* Though the statement of the propositions emanated from Jordanus, the demonstrations and arrangement were Tartaglia's. The statement of a proposition without the proof is of no value. Answering Ferrari's reference to the cubic:† "It would be presumptous of me to give the impression that the result which I discovered could not also have been discovered at other times and by other persons, and that they cannot likewise be discovered in the future and by other persons: even when they will not be given to the public by Signor Hieronimo or myself. But this can I say with truth, that I never saw these things in any author but discovered it myself."

He pokes a thrust at Cardan for not being willing to enter the contest but sending Ferrari instead. And then he dishes up this pretty one:

"You say that you have heard that in the past few years I have made machines and invented several types of instruments and that people think that by my persistent knowledge I have succeeded in making a machine with which I can shoot clear to Milano while I am stationed in Venice.

"Regarding this particular, I answer they are not at all wrong. For since the presentation of your Cartello I have actually built one with which, while I am in Venice, I can shoot, not only as far as Milano,

*Risposta II, pp. 7-8.
†Risposta II, p. 6.

but even as far as Pavia [Cardan had left Milano and was now located in Pavia], and shoot with such a direct aim that I will not only scare you and Signor Hieronimo but cause you great anguish."*

Then he proposes a challenge of thirty one questions.† He states that he can solve them, adding: "I am not like Signor Hieronimo who presents cases he does not know how to solve himself."

Ferrari counters with 31 other questions in the Cartello III (June 1) without answering Tartaglia's. This letter is a highly insulting piece. "In response to my reply," he says, "I have received your *tartagliata* [a pun on the etymology of Tartaglia's name]: which, though long and confused, contains nothing but insults, refusals to admit defeat, and a fixed idea of wanting to fight while running away."

In Risposta III (July 9) Tartaglia gives the solution to Ferrari's 31 questions and boasts he is the victor. In Cartello V (October, 1547) Ferrari tears Tartaglia's answers to pieces mercilessly and claims only five are correct. This Cartello is almost a book in size, all of 55 pages; 41 page are mathematics and contain Ferrari's solutions of Tartaglia's challenge list.

Eight months pass before the answer comes, the longest time between any of these exchanges. And then the unlooked-for happens: on June 1, 1548, Tartaglia accepts the challenge to a public disputation and even to have it in Cardan's and Ferrari's own bailiwick, Milano.‡ How and why the change? He may have become so exasperated with Ferrari's manhandling of his solutions in Cartello V and of other provocative matters in the Cartello—and he did want a duel with Cardan—that he was willing to forego his preferred mode of question lists. Or there may have been other reasons as insinuated by Ferrari, necessitating "Brescia *versus* Milano."§ For city pride and championships did not begin in the 20th century. Tartaglia had recently moved to Brescia where some of its chief citizens had promised him liberal remuneration for giving public lectures on Euclid. Ferrari insinuates that his acceptance of the challenge to Milano was one of the stipulations.

Tartaglia accepted the challenge. But let no one think there was sportsman's etiquette there. Cartello VI (July 14, 1548) and Risponta VI (July 24, 1548) perforce takes up arrangements on the business element. But they have plenty of room for smarting sentences, especially Ferrari's. He seems to have fed on his own anger and vindictiveness.

*Risposta II, p. 11.
†Risposta II, p. 15-20.
‡Risposta V, p. 7.
§Cartello VI, p. 9.

For a year and a half these tirades had continued. On August 10, 1548, the "disputation" took place, and with what outcome we read in Tartaglia's own account, written* nine years later, in an article interpolated in his *General Trattato.*

It follows:

"In 1547 Cardan and his creature Ludovico Ferraro brought me a challenge in two printed pamphlets. I addressed to them 31 questions, with the stipulation that they should be solved in 15 days after receiving them. After that the solution should be considered as not arrived. They let two months pass without giving any sign of existence, and then they sent me 31 questions without giving me the solutions of any of mine; besides, the term stipulated had passed by more than 45 days. I found the solution of 10 of them the same day, the next day a few more, later all the rest, and, so as not to let pass the interval of 15 days, I hurried to get them printed and sent to Milano. In order to conceal their slowness in answering my questions or at least a few of them they took up my time with other matters full of silly foolishness, and it was not till the end of seven months that they sent me a public reply where they boasted that they had solved my questions. However, even had that been true, those solutions given so long a time after the term fixed would have been without any merit; but the greater part of them were completely wrong. I desired to proclaim publicly that they were wrong, so, being in Brescia and hence in the neighborhood of Milano, I sent to them both a printed challenge asking them to meet me the following Friday, August 10, 1548 at 10 10 o'clock near the church called the Garden of the Order of Zoccolante to argue publicly my refutation of their pretended solutions. Cardan, so as not be at the examination, left Milano hurriedly.

"On the day set Ferraro came alone to the meeting-place, accompanied by a crowd of friends and by many others. I was alone with my brother whom I had taken along from Brescia. I went before the entire crowd and began by giving briefly an exposition of the subject for discussion and the reason for my arrival in Milano. When I wanted to take up the refutations of the solutions I was interrupted for a period of two hours by words and actions under pretext that there should be chosen, at that very place, a certain number of judges from the auditors present, all friends of Ferraro and to me entirely unknown. I would not consent to such a trick and said that my understanding was that all the auditors were judges, the same as those who read my refutation when printed. Finally they let me speak, and in order not

*Part II, Book II, Chapter, 7, article 7.

to tire my audience I did not commence with tedious topics from number theory and geometry, but it seemed to me suitable to refute their solution of a question in Chapter 24 in Ptolemy's Geography; and I constrained Ferraro publicly to own that he was in error. When I wished to continue they all began to shout that now I should discuss my own solutions of the 31 questions that had been proposed to me and which I had solved in 3 days. I objected that they should let me finish what concerned my refutations, then I would take up what they asked for. Neither reasoning nor complaining could be heard. They would not let me speak further and gave the word to Ferraro. He began by saying I had not been able to solve the fourth question in Vitruvius, and he expatiated on this clear till the supper hour. Then everybody left the church and went home."

So far Tartaglia. Seeing the temper of the crowd and fearing violence, he did not wait to continue his disputation the next day but hurriedly left for Brescia by another road than that by which he had come, glad to keep his life.

So ended this combat at the Church of Zoccolante, original even for this age.

VIII.

In 1556, (ten years after the appearance of *Quesiti*), came the first two parts of Tartaglia's great life work, for the contents of which he had reserved many discoveries made even before the *Inventioni* of 1546.

It was the *General Trattato di numeri, et misure,* a huge, ambitious and well-written work on arithmetic and algebra. It is said to be the best work on arithmetic in the entire century. The third part, which was not published till 1560, was left uncompleted; for Tartaglia died in 1557. It was largely algebra and it is thought the last division was to have included his work on the cubic equation. As it is, we have only so much of his work in this field as is found in the *Quesiti* of 1546.

The mystic equation:

$$e^{i\pi}+1=0$$

is indeed awe-inspiring. Here, solidly welded together, are the important representatives of the *real,* the *complex,* and *natural* number system.

$i^i=e^{-\pi/2}$, a real!

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

Some Weaknesses in Mathematical Training

By ALAN D. CAMPBELL
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Recently we analyzed the weaknesses of graduate students in mathematics in the United States, the causes of these weaknesses, and their probable cure. We found that these graduate students are often weak in assimilating new ideas and new types of proofs, that they sometimes handle advanced methods in a purely formal manner, that they are frequently blind to problems all around them, that some of them do not show originality or a questioning attitude, that a few of them approach graduate courses with the old attitude of so many pages per assignment. They are often poor in manipulations, they have forgotten so much, perhaps they never really understood the fundamentals, perhaps they have a superficial knowledge of undergraduate mathematics or have memorized too much. They fail to grasp the extensions of old ideas and definitions. They are easily discouraged, they do not study the new material thoroughly enough or go over it again and again, they do not look up the literature and other texts, they do not see why the new ideas are necessary, they do not ask themselves if they can generalize some of the results or how they would go about it, when they read a new theorem they do not stop to think how they would undertake to prove it. They have a sort of compartmental view of mathematics, they are bound by early mathematical habits, they rebel against the more modern and powerful methods.

As to specific weaknesses, to name a few, we found that the graduate students are often poor in handling complex numbers, determinants, matrices, mathematical induction, solid analytics, infinite series, partial derivatives, total differentials, simple differential equations, necessary and sufficient conditions, postulational methods, the idea of a definite integral, the mean value theorems, inequalitles,

inverse trigonometric functions, invariants, transformations of variables, Taylor's expansion for a function of two or more variables, multiple integration, poles and polars, cubic curves, vectors, tangent planes to a surface, families of lines and of cuves, the idea of an irrational number, oblique axes, line integrals, line loci, such coordinate systems as space polar coordinates and parametric coordinates, Euclid's algorithm for the greatest common divisor, the line at infinity.

As to the causes of these weaknesses, we found from the history of mathematics that some are due to the inherent difficulties of the subject, many are due to poor teaching and poor texts, many to conservative and old-fashioned courses, many to the shortening of the time given to the mathematical studies. The causes often go back into high school and the grades, where students frequently just work problems according to a pattern set them, are told not to read the text, have all originality and curiosity stifled (perhaps because the teacher knows so little mathematics or is too busy and overworked).

Let us consider some possible cures. We feel that in the secondary schools some algebra, plane and solid geometry, and trigonometry should come earlier in the grades. Negative and fractional exponents should be used more in high school. Improper fractions should replace mixed numbers. Determinants and a few derivatives of algebraic functions should be taught in high school, and used. Newton's method of approximation should replace Horner's. The maxima and minima should be found in drawing parabolas and other curves. Less arithmetic hsould be given, more handling of algebraic fractions. The teachers should be clear in their own minds as to what items in the courses are important for later use. For example, the quadratic formula should replace much of the factoring, more examples should be done by algebra instead of arithmetic, ratios should be expressed as fractions, in solving simultaneous linear equations elimination of variables should give way to the use of determinants, logarithms and their uses should be stressed. These and other important methods and ideas must be learned early or they never become habitual and familiar.

In college (and also in high school) the artificial barriers between algebra and trigonometry, analytics and calculus we feel should be broken down. Examples from chemistry, statistics, and other applications should be introduced to vitalize the mathematical courses and give the student training in actual uses of the subject. Too often students learn the calculus, for instance, but cannot use it. More explanations of the reasons for the introduction into mathematics of such ideas as the derivative, the integral, and the like should be given in the texts and by the teachers. Again, we must somehow make

the student realize that there are other types of cylinders and cones besides the right circular variety, that quadrilaterals may be re-entrant, that there are other curves besides the conic sections, that tangents may cross these other curves or touch them again, that we can represent a point on the negative part of the x-axis as $(a, 0)$ and do not need to use $(-a,0)$, that a general analytical proof requires letters as coordinates, that the geometry of mathematics is not that of physical space and so may have n dimensions, that the idea of a limit has already been used to discuss the area of a circle and its circumference.

There is a constant pressure downward of the modern ideas and methods of mathematics not only from graduate courses in mathematics but also from other subjects. Matrices and vectors are appearing in psychology, tensors in physics, derivatives and integrals in botany. We must find room for these in our earlier courses in mathematics where they will have time to become thoroughly assimilated. We ought to teach these topics and not leave them for psychologists and others to teach. We must throw out unnecessary topics and details from these earlier courses. For example, we feel we should do less solution of triangles in trigonometry, also curve tracing in analytics should make use of derivatives, matrices should appear in analytics, the area of a triangle and the equation of a line should be used in the determinant form, the idea of a group should come into algebra and analytics, rotations and translations in analytics should be stressed and given also from the viewpoint of keeping the axes fixed and moving the plane, oblique axes and homogeneous coordinates should come somewhere along here, also cross-ratios. Point-sets should appear in advanced calculus, general projective ideas and the plane of the complex variable should be introduced in advanced analytics. In the calculus we waste much time with formal integration and should use the tables of integrals, the definite integral should precede the indefinite integral, the reasons why we can select certain elements of area or volume for integration should be given, infinite series should be presented in reverse order as generalizations of polynomials by starting with series having variable terms, more common sense tests should be used in distinguishing maxima and minima, theory of limits should appear somewhere here or in advanced calculus. There should be a change of emphasis in many courses and more motivation. A purely cultural course in general mathematics should be given for those who do not intend to go on in the subject but want to know "what it's all about." Complete courses in the separate mathematical disciplines are not always necessary. More unified courses should be given, even

courses giving sufficient parts of different branches of mathematics for specific uses such as in statistics or in the sciences. Courses should be given to overcome the above mentioned weaknesses of graduate students of mathematics, such as a course in the fundamental concepts of mathematics, courses in projective geometry, advanced analytics, advanced calculus, algebraic analysis, and the like.

A comparison of the college text-books in mathematics with those in physics, chemistry, botany, zoology, modern languages, will show how conservative and old-fashioned many of our texts and courses are. No wonder mathematics is losing ground as an elective or required study at the very time when more and more mathematics is being used in other subjects. No wonder the teachers of statistics, chemistry, and mechanics, for example, must take time out to teach some algebra and calculus. Mathematics is such an old subject it is ultra-conservative. Many of our research papers are so abbreviated and written according to so artificial and standardized a pattern that they are dull and difficult to read. This fault has crept into our texts and our courses. We deliver so many formal lectures when informal discussion and supplementing of a standard text would be more successful. Much progress has been made in grade school and high school texts and courses. Much is still to be done in revising college texts and courses and in bringing them up to date.

"It seems almost as ungenerous as it is common among inexperienced critics to attribute the failures of pupils to faults in the teaching, and if I join with others in attributing the apparent failure of mathematical teaching in the schools to the nature of the teaching, it is because on looking through accepted textbooks of the fundamental sciences I find no echo of the way in which every science is developed, through patient observation, classification, and induction to the deductive study that is essentially mathematical. I find a careful and logical presentment of the later stages in the development of the subject, interesting to the experienced student looking back over the ground he has traversed, but separated from the beginner by the whole stage that corresponds with the inductive period in the development of a science.

"The teacher and the beginner who is unable to work out for himself the inductive stage move on different planes, and in consequence the student learns his science as he would a foreign language. by the use of his memory and his formulae. He is using the wrong faculties and abhors the study."—By W. N. Shaw, in Glasgow Report, p. 72, as quoted in J. W. A. Young's *The Teaching of Mathematics.*

Mathematical World News

Edited by
L. J. ADAMS

In addition to the usual undergraduate courses the following mixed and graduate courses will be given at the University of Pennsylvania, Summer School, 1938: *Infinite Series and Products,* Professor Mitchell; *Differential Equations* and *Differential Geometry,* Professor Beal; *Diophantine Analysis,* Professor Caris; *Finite Differences,* Professor Shohat.

At the recent meeting of the American Association for the Advancement of Science held at Indinapolis, Professor J. R. Kline was elected Vice-President of the Association and Chairman of Section A for the year 1938.

Dr. J. A. Shohat, University of Pennsylvania, was elected Fellow of the American Institute of Statistics.

The following courses in Mathematics are announced for the Summer Quarter of 1938 at the Texas Technological College in addition to the usual elementary courses: *Elementary Differential Equations, Theory of Equations, Thesis Course in Analysis for the Master's Degree, Advanced Differential Equations,* Professor Michie; *Mathematics of Insurance, Mathematics of Finance,* Professor Langston.

The next meeting of the Wisconsin Section of the Mathematical Association of America will take place on May 14, 1938, at West De Pere, Wisconsin, where the members shall be guests of St. Norbert's College.

The regular meeting of the Southern California Section of the Mathematical Association of America will be held at Pomona College, Claremont, California, on Saturday, March 26, 1938. A program of invited addresses, arranged by the program committee, Professor C. G. Jaeger, chairman, will consist of the following:

1. *Mortality tables—their construction, graduation, and use,* Mr. Alfred L. Buckman, Occidental Life Insurance Co.
2. *Analysis of cosmic rays with the help of terrestrial and solar magnetic fields,* Prof. Paul S. Epstein, California Institute of Technology.

3. *Using the unusual—instructional values in the neglected exceptional cases, with examples,* Prof. S. E. Urner, Los Angeles Junior College.
4. *Stability from the algebraic, analytical, and technical standpoint,* Prof. Harry Bateman, California Institute of Technology.
5. *Some uses of matrix algebra in differential equations,* Prof. Wm. M. Whyburn, University of California at Los Angeles.

The University of Chicago will offer sixteen courses and seminars in mathematics during the coming summer session. Additions to the staff for the summer include Ralph Hull, University of Illinois; Nathan Jacobson, University of North Carolina; and Daniel Dribin, National Research Fellow. A conference on algebra will be held from June 28 to July 1, preceded by six orientation lectures from June 20 to June 27. Invited specialists for the algebra conference include Professors Artin of Notre Dame, Brauer of Toronto, Ingraham and MacDuffee of Wisconsin, Latimer of Kentucky, Lefschetz of Princeton, Schilling, Williamson and Zariski of Johns Hopkins, and Baer of North Carolina. Qualified students and mathematicians are invited to attend this conference.

In addition to the usual undergraduate courses the University of California at Los Angeles will offer: *The Teaching of Mathematics,* Professor E. R. Hedrick; *Elementary Mathematics from an Advanced Standpoint,* Prof. P. H. Daus; *The Calculus of Variations,* Prof. W. M. Whyburn. These are for the summer session, 1938.

"Part of our knowledge we obtain direct; and part by argument. The Theory of Probability is concerned with that part which we obtain by argument, and it treats of the different degrees in which the results so obtained are conclusive or inconclusive.

"In most branches of academic logic, such as the theory of the syllogism or the geometry of ideal space, all the arguments aim at demonstrative certainty. They claim to be *conclusive.* But many other arguments are rational and claim some weight without pretending to be certain. In Metaphysics, in Science, and in Conduct, most of the arguments, upon which we habitually base our rational beliefs, are admitted to be inconclusive in a greater or less degree. Thus for a philosophical treatment of these branches of knowledge, the study of probability is required."—Quoted from J. M. Keynes' *A Treatise on Probability.*

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

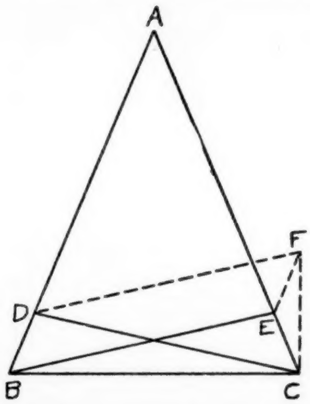
This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

No. 146. Proposed by *A. Blake*, U. S. Coast and Geodetic Survey, D. C.

If two angle bisectors of a triangle are equal, the sides to which they are drawn are equal. Generalize this to the case of n–sectors, i. e., lines which divide the angles in the ratio $n:1$. instead of $2:1$.

Solution by *C. W. Trigg*, Eagle Rock H. S., Los Angeles.



Given the triangle ABC with equal internal n–sectors, BE and CD,which divide the angles so that $\angle CBE : \angle EBA :: \angle BCD : \angle DCA$. Assume that $\angle B > \angle C$, then $\angle EBC > \angle BCD$ so $EC > BD$. Also, $\angle EBD > \angle DCE$. Draw DF parallel to BE and EF parallel to BA, and connect C and F. In the parallelogram $BDFE$, $BE = DF$, $BD = EF$ and $\angle EBD = \angle EFD$. Now since $BE = CD$, triangle FDC is isosceles, whence

$\angle EFD + \angle EFC = \angle DCE + \angle ECF$. Therefore, $\angle EFC < \angle ECF$ and $EC < EF = BD$. Since the assumption leads to two contradictory conclusions, it is not true. Therefore, $\angle B = \angle C$ and the triangle is isosceles.*

No. 194. Proposed by *Walter B. Clarke*, San Jose, California.

Given an isosceles right triangle with $a = b$. Take D on AB so that $AD = a$. Show that if another triangle is constructed with circumradius equal to AD and inradius equal to BD, the circumcenter of the second triangle will lie on its incircle.

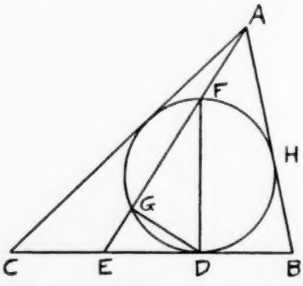
Solution by *C. W. Trigg*, Eagle Rock H. S., Los Angeles.

$AD = a = R$. As a consequence of the Pythagorean Theorem, $BD = a(\sqrt{2} - 1) = r$. By Euler's Theorem, the distance between the circumcenter and incenter of a triangle is given by $d^2 = R(R - 2r)$ $= a[a - 2a(\sqrt{2} - 1)] = a^2(3 - 2\sqrt{2})$. Hence $d = a(\sqrt{2} - 1)$, and the circumcenter of the second triangle lies on its incircle.

No. 195. Proposed by *Walter B. Clarke*, San Jose, California.

Let A be any vertex of a triangle ABC; D and H the contact points of the incircle with sides BC and AB, respectively. E is the contact point of the excircle relative to A with BC. AE cuts the incircle at F and G, in that order. Show that (1) triangles DEG and DFG are similar; (2) $AF : FE :: AH : BC$.

Solution by *C. W. Trigg*, Eagle Rock H. S., Los Angeles.



(1) DF is a diameter of the incircle (No. 121, October, 1936, p. 56,) so FDE is a right triangle, and since angle FGD is inscribed in a semicircle, DG is perpendicular to FE, hence triangle DEG is similar to triangle DFG.

*This is a direct adaptation of the proof for the case $n = 2$ from Altshiller-Court *College Geometry*, page 66.

$$(2)\qquad AE^2 = AB^2 + BE^2 - 2AB\cdot BE\cos B$$

$$= c^2 + (s-c)^2 - (s-c)(a^2+c^2-b^2)/a$$

$$= \frac{s}{a}\left[a(s-a) + (b-c)^2\right].$$

$$FE^2 = FD^2 + DE^2 = (2r)^2 + (b-c)^2$$

$$= 4(s-a)(s-b)(s-c)/s + (b-c)^2$$

$$= \frac{a}{s}\left[a(s-a) + (b-c)^2\right].$$

Hence $$AE : FE :: s : a :: (AF+FE) : FE,$$

so $$AF : FE :: (s-a) : a :: AH : BC.$$

No. 196. Proposed by *Walter B. Clarke*, San Jose, California.

The nine-point circle of a triangle is centered at the midpoint of the line joining incenter and circumcenter. Show that it bisects the three lines joining incenter to vertices.

Solution by *W. T. Short*, Oklahoma Baptist University.

Let O be the circumcenter, I the incenter, and N the nine-point center of the triangle ABC. Let the nine-point circle cut IA, IB, IC at A', B', C', respectively. In the triangles IOA and INA', we have $IO = 2\cdot IN$ and $OA = 2\cdot NA'$ with a common angle at I. Therefore $IA = 2\cdot IA'$. Likewise $IB = 2\cdot IB'$ and $IC = 2\cdot IC'$. Since the nine-point center of any triangle lies midway between orthocenter and circumcenter, this triangle has incenter and orthocenter coincident.

Also solved by *C. W. Trigg*.

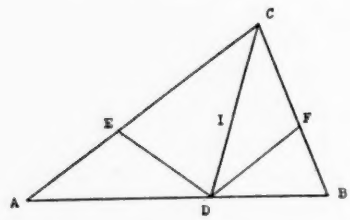
No. 197. Proposed by *Walter B. Clarke*, San Jose, California.

Let I be the incenter of the triangle ABC, and let CI cut AB at D. If E and F lie on CA and CB so that angles

$$EDA = FDB = C/2,$$

will DE equal DF ?

Solution by *Karleton W. Crain*, Purdue University.



In triangle EDA, $\angle AED = 180° - A - C/2.$

In triangle FDB, $\angle DFB = 180° - B - C/2.$

Therefore, $\angle AED + \angle DFB = 180°.$

Now $DE : CD = \sin\dfrac{C}{2} : \sin\angle DEC = \sin\dfrac{C}{2} : \sin\angle DFB.$

and, $DF : CD = \sin\dfrac{C}{2} : \sin\angle DFC = \sin\dfrac{C}{2} : \sin\angle DFB.$

Therefore, $DE = DF.$

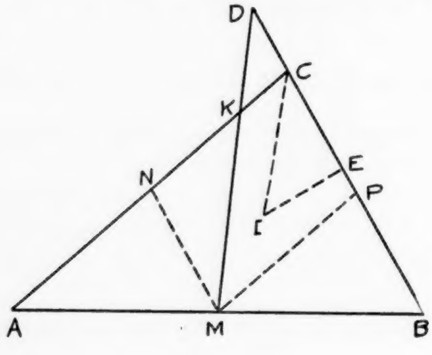
Also solved by *W. T. Short* and *C. W. Trigg*,

No. 198. Proposed by *Walter B. Clarke*, San Jose, California.

Given the triangle ABC with $a < b < c$. Let I be the incenter and M the midpoint of AB. D is taken on BC prolonged so that DM bisects the perimeter of the triangle. Let E be the contact point of the incircle with BC. Show that

$$CI : CE :: CA : DM.$$

Solution by *C. W. Trigg*, Eagle Rock High School, Los Angeles.



Let N and P be the mid-points of AC and BC respectively. Then $MP=\frac{1}{2}b$, $MN=\frac{1}{2}a$, $\angle ANM=\angle C$, and since their sides are parallel, $\triangle MNK \sim \triangle DPM$. Now if DM bisects the perimeter at M and K then $AK=s-\frac{1}{2}c=\frac{1}{2}(a+b)$ so $NK=\frac{1}{2}a$. Hence $\triangle NKM$ is isosceles and $\angle NKM=\angle NMK=\frac{1}{2}C=\angle MDP$. $\angle MNK=A+B=\angle DPM$. In the triangle DPM,

$$DM=\frac{b\sin(A+B)}{2\sin\frac{1}{2}C}=\frac{b\sin C}{2\sin\frac{1}{2}C}=\frac{2b\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin\frac{1}{2}C}=b\cos\tfrac{1}{2}C.$$

$$CA=b \text{ and } CE/CI=\cos\tfrac{1}{2}C.$$

Therefore $$CI:CE::CA:DM.$$

No. 201. Proposed by *G. W. Wishard*, Norwood, Ohio.

Show that:

(1) Every perfect odd octonary cube ends with the same figure as the root.

(2) Every perfect even octonary cube ends with 0.

(3) Any number of zeros may be annexed to any perfect octonary cube, and the result will be a cube.

Solution by *C. W. Trigg*, Eagle Rock High School, Los Angeles.

In any scale of notation the terminal digit of a product is the same as the terminal digit of the product of the terminal digits of the factors. So in the scale of 8 it is only necessary to list and examine the cubes of the digits, i. e. $0^3=0$, $1^3=1$, $2^3=10$, $3^3=33$, $4^3=100$, $5^3=175$, $6^3=330$, $7^3=527$. (1) and (2) are now evident. Since $2^3=10$, annexation of k zeros to N^3 is the same as multiplying N^3 by 2^{3k}. (3) is then true since $N^3 2^{3k}$ is a perfect cube.

Also solved by the *Proposer*.

No. 204. Proposed by *V. Thébault*, Le Mans, France.

In what system of numeration will there exist two perfect squares of four digits:

$$aa44=(cc)^2, \quad 44aa=(dd)^2,$$

given that, in the decimal system, a is formed of two equal digits?

Solution by *F. G. Fender*, New Brunswick, New Jersey.

If B is the desired base, the two given equations may be put

$$(1) \qquad (B+1)(aB^2+4)=c^2(B+1)^2, \quad (B+1)(4B^2+a)=d^2(B+1)^2.$$

From the first of these we may write

$$c^2(B+1)=aB^2+4=(B+1)[(a-1)B+(B-a+1)+(a+3-B)/(B+1)].$$

Thus $a+3-B=0$ and $a=B-3$ and $c=B-2$. Put now this value of a into the second of equations (1) to obtain $d^2=4B-3$. If the odd number d be replaced by $2m+1$, we obtain $B=m^2+m+1$ and $a=m^2+m-2=(m+2)(m-1)$. Finally a will be a multiple of 11 and less than 100 only when $m=9$. Then B is 91 and we have the unique solution $\overline{88}\ \overline{88}\ 4\ 4=(\overline{89}\ \overline{89})^2$, $4\ 4\ \overline{88}\ \overline{88}=(\overline{19}\ \overline{19})^2$.

Also solved by *C. W. Trigg*, *G. W. Wishard* and the *Proposer*.

No. 205. Proposed by *A. Gloden*, Luxembourg.

What is the general solution in positive integers of the following diophantine systems:

$$\text{I.}\begin{cases} x+ay=u^2 \\ x^2+ay^2=v^2 \end{cases} \qquad \text{II.}\begin{cases} x+ay=u^2 \\ x^2+bxy+ay^2=v^2 \end{cases}$$

$$\text{III.}\begin{cases} x+y=u^2 \\ x^3+y^3=v^2 \end{cases} \qquad \text{IV.}\begin{cases} x+y+z=u^2 \\ x^2+y^2+z^2=v^2 \end{cases} ?$$

Solution by the *Proposer*.

The general solution of an equation of the form $x^2+bxy+ay^2=z^2$ is given by $x=m^2-an^2$, $y=2mn+bn^2$ and $z=m^2+bmn+an^2$. For I, II or III, after determining x and y to satisfy the second equation and substituting these values into the first we obtain an equation in m, n and u which may be solved by a repetition of the same process. Thus, for example in I, we use $x=m^2-an^2$, $y=2mn$ and $v=m^2+an^2$ to obtain $m^2+2amn-an^2=u^2$. The solution of this last equation is given by $m=r^2+as^2$, $n=2rs+2as^2$ and $u=r^2+2ars-as^2$, which then yields a two parameter solution of the original system.

To solve the second equation in IV, we employ the identity

$$(7m^2-12mn+7n^2)^2=(6m^2-14mn+6n^2)^2+(3m^2-3n^2)^2+(2m^2-2n^2)^2$$

to give the values of x,y and z which reduce the first equation to a form which may be solved as indicated above.

Editor's Note. The solution for I, II and III above will be perfectly general if the values of x, y and z which satisfy $x^2+bxy+ay^2=z^2$ are multiplied by an arbitrary constant, which need not be an integer.

See L. E. Dickson, *Introduction to the Theory of Numbers*, (1929), p. 46 and p. 48, ex. 6. The solution of IV lacks generality in a more serious manner. It was apparently obtained from the solution of $x^2+aY^2=v^2$ by assuming $y=3Y$ and $z=2Y$, $a=13$. Evidently we may obtain other solutions by use of $y=pY$, $z=qY$, with p and q unrestricted. The most general solution of $x^2+y^2+z^2=v^2$ is given in Carmichael, *Diophantine Analysis*, (1915) pp. 35-43. When these values are substituted back into IV, however, they introduce a difficult further problem.

Late solution: No. 129, *C. W. Trigg.*

PROPOSALS

No. 227. Proposed by *Albert Farnell*, Centenary College.

A perfectly elastic ball is projected from a certain height above the horizontal plane at an angle of 0° and with an initial velocity of 60 feet per second. If it strikes the plane at an angle of 60° and rises to two-thirds the preceding height with each bounce, find the total area covered in the plane in which it moves, disregarding air resistance.

No. 228. Proposed by *Walter B. Clarke*, San Jose, California.

Two straight lines XX' and YY', intersect at 0 so that angle XOY is acute. Q is a variable point on XX' and P a fixed point on OY. The bisector of angle PQX will cut YY' at points outside a certain segment ST of the line. For what angle XOY will $PS=PT$?

No. 229. Proposed by *Walter B. Clarke*, San Jose, California.

Locate the vertices of a square inscibed to a given triangle so that one side of the square shall be coincident with one side of the triangle, produced if necessary.

No. 230. Preposed by *V. Thébault*, Le Mans, France.

If a plane cuts the surface and volume of a tetrahedron into two equivalent parts, this plane passes through the center of the inscribed sphere of the tetrahedron.

No. 231. Proposed by *V. Thébault*, Le Mans, France.

If the circumcircle and the nine-point circle of a triangle ABC intersect at an angle of 60°:

(1) the area of the orthic triangle $A'B'C'$ is equal to half the area of ABC;
(2) one of the tritangent circles of the triangle $A'B'C'$ is equal to the nine-point circle of ABC;
(3) the line GK which joins the barycenter to the Lemoine point of the triangle ABC is parallel to the Euler line of the triangle $A'B'C'$;
(4) the polar line of G with respect to the polar circle of triangle ABC* passes through the center of the nine-point circle;
(5) the circumcircle and the polar circle cut each other on the nine-point circle;
(6) the Lemoine point K and the poles of the sides BC, CA, and AB, with respect to the circumcircle lie on the polar circle;
(7) if H is the orthocenter and m_a, m_b, m_c the lengths of the medians of the triangle ABC, then $AH : BH : CH :: m_a : m_b : m_c$.

No. 232. Proposed by *V. Thébault*, Le Mans, France.

In the system of numeration with base 11, form a perfect square of six digits having each of the forms *abcabc* and *aabbcc*.

No. 233. Proposed by *E. P. Starke*, Rutgers University.

Find all sets of four distinct integers, having no common divisor greater than 1 and such that each is a divisor of the sum of the other three.

No. 234. Proposed by *D. C. Duncan*, Compton Junior College, California.

(1) If the base and side of an isosceles triangle are relatively prime integers, prove that the internal bisectors of the base angles cannot be integers.
(2) Find the smallest integral isosceles triangle whose three internal bisectors are all integers.
(3) Obtain all triangles whose three sides and three internal bisectors are simultaneously rational.

No. 235. Proposed by *C. E. Springer*, University of Oklahoma.

The direction of a gun on a horizontal plane may lie anywhere within a cone of semi-vertical angle Θ, the axis of the cone being fixed

*See Johnson's *Modern Geometry*, p. 176.

at an angle α with the horizontal. If the initial speed of the projectile (v_0) is constant, find the equation of the curve which bounds the field of fire on the horizontal plane. Show that according as

$$\theta \gtreqless \left| \frac{\pi}{2} - 2\alpha \right|,$$

there is a bi-tangent to the curve perpendicular to the vertical plane of fire, or a tangent to the curve at a point in the vertical plane of fire, or no tangent to the curve perpendicular to the plane of fire.

Note. The case in which θ is so small that its powers higher than the first can be neglected, is given by MacMillan, *Theoretical Mechanics*, (1927), exercise 16, page 263.

HALLEY'S COMET

(Tune: "Sally in Our Alley")

Of all the meteors in the sky
There's none like Comet Halley,
We see it with our naked eye
And periodically.
The first to see it was not he,
But still we call it Halley;
The notion that it would return
Was his originally.

Of all the years we've lately seen
There's none to rival this year.
Because though busy we have been
We're likely to be busier.
When five and twenty years are out
We look for Comet Halley:
He said it would come back again,
Perhaps perpetually.

We probe the secrets of the sun,
And most effectually;
There's much good honest work been done
Selenographically.
Whatever quest may prove the best
We all revere bold Halley,
Who said his comet would return,
And mathematically.

—*by Prof. H. H. Turner, FRS.,*
Pres. Math. Ass'n 1910.

Reviews and Abstracts

Edited by
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Grosse Mathematiker. By Gerhard Kowalewski. J. F. Lehmanns Verlag, Berlin & Munich, 1938. 300 pages, with 35 figures and 16 portraits. Price 8.70 marks.

This volume has its origin in the lectures which the author gave at the universities of Leipzig, Greifswald, Bonn and Prague, and in which he made it a point to stress the historical side in the teaching of mathematics. In this, as far as the material presented is concerned, Kowalewski has succeeded very well. He has written a book which makes interesting reading and stimulates the reader to take greater interest in the history of mathematics.

When it comes to the selection of the material itself, the reviewer must confess that he is rather disappointed. It seems to me that the list of "great mathematicians" in the book is lacunary and contains some names whose inclusion would seem questionable. As examples I mention Moschopulos, Rhaticus, and Faulhaber. Again, if it is eminently proper to have men like Euler, Lagrange, Gauss, etc., listed among great mathematicians, it is doubtful if any one would be willing to put Pfaff on a similar level. If the author had stated that his list was mostly restricted to representatives of analysis in the 18th and 19th centuries, one could understand omissions of names like Poncelet, Steiner, Clebsch, etc.

From the standpoint of the critical reader, sentences like these: "Man sagt" = it is said; "man ist der Meinung" = one is of the opinion, are not apt to increase confidence in a statement. For example, Kowalewski says: "One is of the opinion, that the proof given in Euclid of of the Theorem of Pythagoras, comes from Pythagoras himself." This is not true.

On the other hand it is commendable that the author mentions Lobatschewsky in the first line as one of the founders of non-Euclidean geometry. In general, it must be said that the author has been fair in his appraisal of the really great mathematicians and has given us a vivid picture of some mathematicians and their work.

University of Illinois. ARNOLD EMCH.

The Romance of the Calendar. By P. W. Wilson. W. W. Norton and Co., Inc., New York, 1937. viii+319+24 pages.

The author tells in a fascinating way the story of the calendar, "the one activity in civilization that can never be out-of-date," an activity so familiar to the individual and society that it is usually taken for granted. He divides the story into five parts with the following titles:

Part I. Background of Calendar History.
Part II. Chronological Succession.
Part III. The Broader Horizon.
Part IV. Human Yardstick.
Part V. The Future.

The searchings of archaeologists disclose that homo sapiens, whose existence was punctuated with the recurring phenomena of light and darkness, was not long in realizing the value of what we call time, and considered it as something to be measured and determined. "There is no document known among men or conceivably lurking in buried cities that can begin to rival in antiquity these attempts to measure time. That measurement may be regarded as man's first cultural thought."

The measurement of time, its division into suitable periods, the enumeration, identification and adjustment of these periods has been a long, slow process beginning thousands of years before Moses gave the Ten Commandments to the people of Israel and continuing to the present time, when the calendar reformer is seeking to adapt the measurement of time to the demands of our complicated and closely knit civilization.

The calendar is derived from the sun, moon and stars, and the writer recounts the many battles waged to determine whether the sun or the moon should be the controlling authority. As early as 4500 years ago there appear to have been local calendars in Babylon. About 3000 B. C. the mind of Babylon and the mind of Egypt came to grips with the same problem of chronometry—that of reconciling a lunar year of 354 days with a solar year of 365 days. In both these countries the moon first held sway but ultimately the sun triumphed. Later, Greece and Rome were confronted by the same problem, and out of that struggle came in 47 B. C. the Julian Calendar and later, in 1577, the Gregorian adjustment. The author carefully details the problems which were solved by these calendars and gives an account of the discovery, in 1936, of the Mayan calendar stone, which dates from the first or second century A. D. It is probably the oldest calen-

dar in the Americas and agrees precisely with the European chronology except for a discrepancy of about ten days due to an error in the Julian calendar.

Today a struggle is going on in India where fourteen important calendars are in use: but in China there is still in force a calendar that is two centuries older than Abraham, fifteen centuries older than Homer and eighteen centuries older than Buddha and Confucius.

A description is given of different mechanical devices for measuring time—various types of sundials, fire clocks, water clocks and sand-glasses—as well as of the ways in which man has arranged his days in weeks and subdivided them into hours, minutes and seconds without astronomical significance.

The book closes with a careful discussion of proposed reforms in the calendar together with the advantages and disadvantages of each. It is suggested that there are five fields of inquiry into the possibility of calendar reform:

"First, we have the months. Are they the most convenient and logical months that can be arranged?

Secondly, we have the weeks. Do they fall as logically and conveniently within the months and the year as might be arranged?

Thirdly, there is the year. Admit that there is nothing wrong with its length. Does it start and end on the best date?

Fourthly, there are the hours of the day. In that limited but important arena is any useful adjustment to be suggested?

Fifthly, there is Easter. Is it necessary in these days that Easter should be a movable feast?"

A study of this part of the book, "The Future", will afford anyone an excellent idea of the defects in our present calendar and of the ways of correcting them.

The Romance of the Calendar can be read with understanding by intelligent laymen who are unfamiliar with mathematics and astronomy, and it will enlarge their appreciation of the contributions made by astronomers, mathematicians, archaeologists, statesmen and churchmen to the development of the calendar. We all owe a debt of gratitude to the author for presenting so much informative material in such an alluring way.

Iowa State College. MARIAN E. DANIELLS.

Plane Trigonometry. By William Kelso Morrill. Farrar and Rinehart, New York, 1938.

This is a brief trigonometry. The text covers 141 pages. In the preface an outline is given, showing the number of lessons that may be devoted to each chapter. This schedule shows that the book can be completed in thirty lessons. There are more than a thousand problems, graded according to difficulty.

General angles are introduced first, rather than acute angles, as is the case in many trigonometry textbooks. Chapter III is called Numerical Computations, and the topics included are the functions of 0°, 30°, 45°, 60°, 90°, angles greater than 90°, use of the natural function tables, and solution of the right triangle. The next chapter is a very clearly written exposition of logarithms and the use of tables of logarithms.

The subject of identities is treated very briefly, the chapter being largely a collection of problems and a list of fundamental identities. On page 66 we find this statement: "It is suggested that when the student proves the following identities, in this and later sections, he does so by working only with one side of the identity and reducing it to the other side." Then, near the bottom of the same page we have this illustrative example:

$$\pm\sqrt{\frac{1-\sin A}{1+\sin A}}=\sec A-\tan A$$

proved by first *squaring both sides.*

The remaining chapters are named The General Triangle, Trigonometric Equations and the Inverse Trigonometric Functions, and Graphs.

The appendix includes a list of formulas and answers to the odd-numbered problems. There are no tables in the book.

Those who give a brief course in trigonometry would do well to examine this text.

Santa Monica Junior College. L. J. ADAMS.

Advanced Calculus. By W. Benjamin Fite. Macmillan Co., New York, 1938. xii+399 pages.

This book is designed to serve as an introductory course in mathematical analysis. The first five chapters cover rigorously the usual material of a first course in calculus; real numbers, functions of one

variable, functions of several variables, Taylor's expansion and the definite integral with the addition of such topics as Euler's theorem on homogeneous functions, Lagrange multipliers, first and second mean value theorem, Euler's constant and the determination of the error involved in Simpson's rule. The next two chapters cover indefinite integrals and improper integrals with applications to elliptic integrals, differentiation and integration under the integral sign, the Beta function, the Gamma function and Stirling's formula. The following chapter takes up double and triple integrals, line integrals, and such theorems as those of Green and Stokes. Chapters IX, X, XI are devoted to infinite series (simple and double) and Fourier series with discussions of the uniform convergence tests of Weierstrass and Abel, infinite integrals of infinite series, quasi-uniform convergence, Dirichlet integrals and the Fourier integral, in addition to the usual topics. The remaining chapters deal with implicit functions (including Jacobians), geometric applications (curves and surfaces) and short introductions to the calculus of variations and to functions of a complex variable.

The choice of the material on infinite series and operations with infinite integrals involving a parameter, and its presentation, make the text very useful both as a reference and as a class book. It is the belief of the reviewer that it attains the aim of the author as stated in the preface. There is a large number of interesting problems. Trivial misprints can be found easily on pages 26, 37, 68, 100, 104, 144, 152, 163, 183, 185, 189, 292, 383. On page 104 pressure is used incorrectly instead of force.

Virginia Military Institute. W. E. BYRNE.

"Now mathematicians are everywhere aware that their science has a most pronounced habit of producing its most important results without seeking permission anywhere else, without ever asking 'by your leave'. Indeed, so striking is this characteristic that at least once in a generation something is produced by their science which causes mathematicians themselves to view with alarm what is happening. It was true in the days of Pythagoras: and whether we consider the appearance of minus, zero and the imaginaries, the appearance of the non-Euclidean geometries of the last century, or that of the non-Riemannian geometries of this century, it is always the algebraic, the geometric, the analytic in its own right that is found forcing its way forward, and coming to prevail despite all clamor."—By Arthur F. Bentley in *Linguistic Analysis of Mathematics.*